Some inequalities for distance-regular graphs (Algebraic Combinatorics)

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1 Definitions

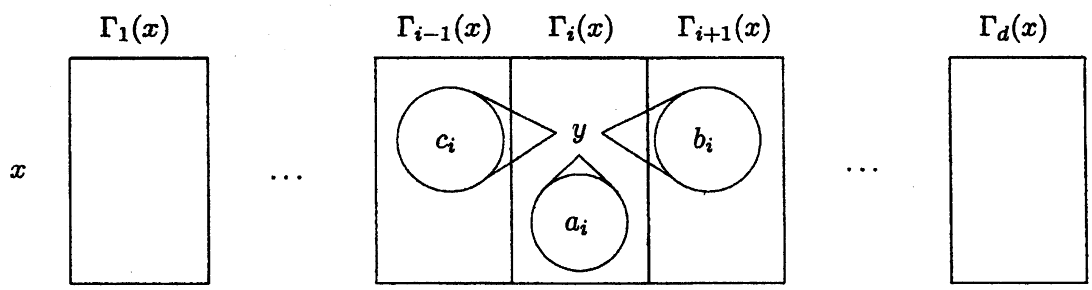
All graphs considered are undirected finite graphs without loops or multiple edges. Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph with usual shortest path distance ∂_Γ. The *diameter* of Γ, denoted by d, is the maximal distance of two vertices in Γ. Let $u \in \Gamma$. We denote by $\Gamma_j(u)$ the set of vertices which are at distance j from u. For $x, y \in V\Gamma$ with $\partial_\Gamma(x,y) = i$, let

$$\begin{array}{rcrcl} C_i(x,y) & = & \Gamma_{i-1}(x) & \cap & \Gamma_1(y), \\ A_i(x,y) & = & \Gamma_i(x) & \cap & \Gamma_1(y), \\ B_i(x,y) & = & \Gamma_{i+1}(x) & \cap & \Gamma_1(y). \end{array}$$

A connected graph Γ is said to be *distance-regular* if

$$c_i = |C_i(x,y)|, \quad a_i = |A_i(x,y)| \quad \text{and} \quad b_i = |B_i(x,y)|$$

depend only on $i = \partial_\Gamma(x,y)$ rather than individual vertices.



The numbers c_i, a_i and b_i are called the *intersection numbers* of Γ. In particular, $k := b_0$ is called the *valency* of Γ. The array

$$\iota(\Gamma) = \left\{ \begin{array}{ccccccc} * & c_1 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_i & \cdots & b_{d-1} & * \end{array} \right\}$$

is called the *intersection array of* Γ. Define

$$r = r(\Gamma) := \max\{ i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1) \}.$$

Let $\Gamma = (V\Gamma, E\Gamma)$ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Define $R_i := \{ (x,y) \mid \partial_\Gamma(x,y) = i \}$. Then $(V\Gamma, \{R_i\}_{0 \leq i \leq d})$ is an association scheme of class d and

$$p^t_{i,j} := \#\{z \mid \partial_\Gamma(x,z) = i,\ \partial_\Gamma(y,z) = j\},$$

where $\partial_\Gamma(x,y) = t$.
Conversely, for an *association scheme* $(X, \{R_i\}_{0 \leq i \leq d})$ *satisfying P-polynomial condition*, the graph (X, R_1) is a distance-regular graph. (See [1] and [3].)

2 The Odd graph and the doubled Odd graph

Let m be a positive integer and X a set of $2m+1$ elements. For $0 \leq t \leq m$ define

$$X_t := \{Y \subseteq X \mid \#Y = t\}.$$

The *Odd graph* O_{m+1} is a graph with

$$V\Gamma = X_m, \qquad E\Gamma = \{(Y, Y') \mid Y \cap Y' = \emptyset \}.$$

The *doubled Odd graph* $2O_{m+1}$ is a graph with

$$V\Gamma = X_m \cup X_{m+1} \quad E\Gamma = \{(Y,Z) \mid Y \in X_m,\ Z \in X_{m+1},\ Y \subset Z\}.$$

Then Odd graph O_{m+1} is a distance-regular graph of diameter m with :

Case : $m = 2t$,

$$\left\{\begin{array}{cccccccc} * & 1 & 1 & 2 & 2 & \cdots & t & t \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & t+1 \\ 2t+1 & 2t & 2t & 2t-1 & 2t-1 & \cdots & t+1 & * \end{array}\right\}.$$

Case : $m = 2t-1$,

$$\left\{\begin{array}{cccccccc} * & 1 & 1 & 2 & 2 & \cdots & t-1 & t \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & t \\ 2t & 2t-1 & 2t-1 & 2t-2 & 2t-2 & \cdots & t+1 & * \end{array}\right\}.$$

The doubled Odd graph $2O_{m+1}$ is distance-regular of diameter $2m+1$ with :

$$\left\{\begin{array}{cccccccccc} * & 1 & 1 & 2 & 2 & \cdots & m-1 & m & m & m+1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ m+1 & m & m & m-1 & m-1 & \cdots & 2 & 1 & 1 & * \end{array}\right\}.$$

More information for these graphs can be found in [3, §9.1.D].

These graphs had been characterized by some of their intersection numbers. In particular, Ray-Chaudhuri and Sprague [8], Cuypers [4, Theorems 4.6-4.7] and Koolen [7, Theorem 16] proved the following result.

Theorem 1 *Let Γ be a distance-regular graph of diameter $d \geq 5$ such that $c_1 = c_2 = 1$, $c_3 = c_4 = 2$ and $a_1 = \cdots = a_4 = 0$. Then Γ is either the Odd graph or the doubled Odd graph.* ∎

3 Some Inequalities for distance-regular graphs

The following are well known basic properties of the intersection numbers.

(1) $1 = c_1 \leq c_2 \leq \cdots \leq c_{d-1} \leq c_d \leq k$.

(2) $k = b_0 \geq b_1 \geq \cdots \geq b_{d-2} \geq b_{d-1} \geq 1$.

(3) $c_i \leq b_j$ if $i+j \leq d$.

We recall the proof of these properties:
Let $(x_0, x_1, \ldots, x_d)$ be a path of length d such that $\partial_\Gamma(x_0, x_d) = d$. Then we have

$$C_1(x_1, x_0) \subseteq C_2(x_2, x_0) \subseteq \cdots \subseteq C_{d-1}(x_{d-1}, x_0) \subseteq C_d(x_d, x_0) \subseteq \Gamma_1(x_0),$$

$$B_0(x_0, x_0) \supseteq B_1(x_1, x_0) \supseteq \cdots \supseteq B_{d-2}(x_{d-2}, x_0) \supseteq B_{d-1}(x_{d-1}, x_0) \supseteq \{x_1\}$$

and

$$C_i(x_0, x_i) \subseteq B_j(x_{i+j}, x_i).$$

∎

The **idea** of this proof is : " Let $X \subseteq Y$ be subsets of $V\Gamma$. Then $\#X \leq \#Y$. "
We prove several inequalities for intersection numbers by applying this idea. For example:

Theorem 2 *Suppose* $1 \leq c_{t-1} < c_t$. *Then* $c_{2t-1} \neq c_t$.

Sketch of the Proof.

Suppose $c_{2t-1} = c_t$ and derive a contradiction. Let $u, x, y \in V\Gamma$ such that

$$\partial_\Gamma(u, y) = 2t-1,\ \partial_\Gamma(u, x) = t-1 \quad \text{and} \quad \partial_\Gamma(x, y) = t.$$

Then there exist $v \in C_{2t-1}(y, u) \setminus C_{t-1}(x, u)$ and $u' \in \Gamma_1(v) \cap \Gamma_{t-1}(x)$ such that

$$B_{t-1}(u', x) \subseteq B_{t-1}(u, x) \setminus C_{t-1}(y, x).$$

Then this implies $b_{t-1} \leq b_{t-1} - c_{t-1}$ which is a contradiction. ∎

Theorem 3 *Let Γ be a distance-regular graph with:*

$$\left\{\begin{array}{ccccccccc} * & 1 & \cdots & 1 & 2 & \cdots & 2 & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \\ k & k-1 & \cdots & k-1 & k-2 & \cdots & k-2 & & \end{array}\right\}.$$
$$\qquad \underbrace{\qquad\qquad\qquad}_{r} \qquad \underbrace{\qquad\qquad\qquad}_{s}$$

Then $s \leq \dfrac{r+4}{3}$.

Sketch of the Proof.

Suppose $s > \dfrac{r+4}{3}$ and derive a contradiction. Let $u, v, x \in V\Gamma$ such that

$$\partial_\Gamma(u,v) = r+1, \qquad \partial_\Gamma(u,x) = 1, \qquad \partial_\Gamma(x,v) = r.$$

Then we can take a path $(u = y_0, y_1, y_2, \ldots, y_{r+1} = y)$ of length $r+1$ such that

$$C_{r+1}(v,y) \subseteq C_{r+1}(u,v) \setminus C_r(x,v).$$

This is a contradiction. ∎

We have the following result by Theorem 1 and 3.

Corollary 4 *Let Γ be a distance-regular graph as in theorem. If $s = r \geq 2$, then $r = 2$ and Γ is either the Odd graph or the doubled Odd graph.* ∎

Theorem 5 *Let Γ be a distance-regular graph with*

$$\left\{\begin{array}{ccccccccc} * & 1 & \cdots & 1 & c & \cdots & c & & \\ 0 & a_1 & \cdots & a_1 & a & \cdots & a & \cdots & \\ k & b_1 & \cdots & b_1 & b & \cdots & b & & \end{array}\right\}.$$
$$\qquad \underbrace{\qquad\qquad\qquad}_{r} \qquad \underbrace{\qquad\qquad\qquad}_{s}$$

Suppose $s = r \geq 2$. Then $a = a_1 = 0$ and $r = 2$. In particular, Γ is either the Odd graph, the doubled Odd graph or the doubled Grassmann graph. ∎

Proposition 6 *Let Γ be a distance-regular graph of diameter $d \geq 2$ and $k \geq 3$. Let q, h be positive integers with $q + h \leq d$.*
(1) If $c_q < c_{q+1}$ and $a_q = 0$, then $c_h \leq c_{q+h} - c_m$ and $b_{q+h} \leq b_h - c_q$.
(2) If $b_{q+h} < b_{q+h-1}$ and $a_{q+h} = 0$, then $c_h \leq b_q - b_{q+h}$.

Sketch of the Proof of (1).

Let $u, x, y \in V\Gamma$ such that

$$\partial_\Gamma(u,x) = h, \quad \partial_\Gamma(x,y) = q \quad \text{and} \quad \partial_\Gamma(u,y) = q + h.$$

Define

$$W := \bigcup_{z \in C_h(u,x)} \{C_{q+1}(z,y) \setminus C_q(x,y)\}.$$

Then

$$\#C_h(u,x) \leq \#W \quad \text{and} \quad W \subseteq C_{q+h}(u,y) \setminus C_q(x,y).$$

Hence $c_h \leq c_{q+h} - c_q$. ∎

4 A characterization of O_k and $2O_k$

Theorem 7 *Let Γ be a distance-regular graph of diameter $d \geq 5$ valency $k \geq 3$ and $r = \max\{i \mid (c_i, b_i) = (c_1, b_1)\} \geq 2$. Suppose one of the following conditions holds. Then Γ is either the Odd graph O_k, or the doubled Odd graph $2O_k$.*

(i) *$a_{m+r} = 0$ and $1 + c_m = c_{m+r} \leq k - 2$ hold for some m with $r \leq m \leq d - r - 1$.*

(ii) *$a_m = 0$ and $2 \leq b_{m+r} = b_m - 1$ hold for some m with $r \leq m \leq d - r - 1$.*

Sketch of the Proof of (i).

We have $a_{m+r} = \cdots = a_1 = 0$ and $c_{r+1} > c_r$. Since $c_m = c_{m+r} - c_r$, the equality holds in Proposition 6. Then we obtain that $c_{m-1} = c_{m+r-1} - c_r$ holds.

Inductively, we have $c_j = c_{j+r} - c_r$ for all $j < m$. This implies that

$$c_{r+1} = \cdots = c_{2r} = 2.$$

The desired result is proved by Corollary 3. ∎

The reader is referred to [2, 5, 6] for more detailed proofs of the results.

References

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